THE STABILITY OF NONISOTHERMAL PLASMA FLOW IN A FLAT MHD CHANNEL

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Papers [1, 2] were devoted to questions of the stability of the laminar flow of a conducting fluid in a transverse magnetic field with Hartmann flow. It was assumed in these papers, however, that the transport coefficients are quantities independent of the flow characteristics; in particular, the temperature and the effect of energy dissipation were not taken into account. When these factors are allowed for it turns out that even for relatively small subsonic velocities, when the medium may be regarded as incompressible, the temperature distribution exerts a considerable influence on the dynamic flow characteristics. Papers [3, 4] deal with this type of flow in an MHD channel which will be called nonisothermal in what follows. It has been shown that under specific conditions the velocity profiles are grossly deformed, and nonmonotonic profiles with inflection points may even appear.

However, the influence of nonisothermal flow on stability is not confined to an alteration of the stability criteria as a result of the change in the velocity profile. When energy dissipation and the fact that the transport coefficients are not constant are taken into account new "dissipative" instability branches appear, as, for example, the overheat instability [5, 8]. This article considers the problem of the hydrodynamic stability of a nonisothermal plasma flow in constant crossed electric and magnetic fields in a flat channel with dielectric walls. The system of equations derived in this paper for the perturbations does, of course, take into account all the instability mechanisms mentioned above, but is difficult to solve. The general system of equations may be investigated in two limiting cases corresponding to the overheat and hydrodynamic instabilities.

1. Initial Steady State. Let the x axis be in the direction of the flow and the y axis be in the direction of the external magnetic field B_0, while the z axis is in the direction of the constant electric field E. The channel is bounded by dielectric walls situated at $y = \pm l$, while the distance between the electrodes in the z direction and the length of the channel in the x direction are assumed to be quite large. It is also assumed that $\omega_e \tau_e \ll 1$, i.e., scalar magnetohydrodynamics may be used. The transport coefficients as functions of temperature are approximated by the following power laws:

$$\sigma = \sigma_0 \left(\frac{T}{T_0}\right)^\alpha, \quad \varkappa = \varkappa_0 \left(\frac{T}{T_0}\right)^\beta,$$

$$\eta = \eta_0 \left(\frac{T}{T_0}\right)^\gamma, \quad (\alpha, \beta, \gamma - \text{const}), \tag{1.1}$$

which is permissible if these parameters are monotonic functions of the temperature of the medium, and the temperature within the MHD channel undergoes only limited changes. Assuming that all quantities are functions of y only we arrive at the following system of equations describing the initial steady state:

$$p + \frac{d}{dy}\left(\eta \frac{dU}{dy}\right) - jB_0 = 0, \qquad \frac{dB_x}{dy} = -\mu j,$$

$$\frac{d}{dy}\left(\varkappa \frac{dT}{dy}\right) + \eta \left(\frac{dU}{dy}\right)^2 + \frac{j^2}{\sigma} = 0,$$

$$j = \sigma (E + UB_0) \tag{1.2}$$

which must be solved for the boundary conditions

$$U(\pm l) = 0, \; T(\pm l) = T_0. \tag{1.3}$$

The system of Eqs. (1.1)–(1.3) may only be solved numerically in the general case. This was done in paper [4], and the results will be employed in what follows. Since we are unable to discuss the properties of the steady state solution in detail, we shall merely remark that the simplest formulation of the problem was chosen when the heat dissipated in the channel passes out through the walls which are maintained at the same conditions. The solution of the steady state problem for U, T, and j depends on six dimensionless parameters: α, β, γ, K, M, N, i.e., it is of the form

$$U = U(y; \alpha, \beta, \gamma, K, M, N)$$

$$\left(M = B_0 l \sqrt{\frac{\sigma_0}{\eta_0}}, \; K = -\frac{E\eta_0}{l^2 p B_0}, \; N = \frac{l^4 p^2}{\varkappa_0 \eta_0 T_0}\right).$$

Here M is the Hartmann number, K is the dimensionless electric field, and N is a thermal parameter. The induced magnetic field B_x depends on the magnetic Reynolds number R_m in addition to these parameters.

2. The Linearized Equations for Small Perturbations. In accordance with Squire's theorem [9] one is usually confined in hydrodynamics to considering two-dimensional perturbations which determine the least value of the critical Reynolds' number. Although Squire's theorem in magnetohydrodynamics has been proved only for flow in a longitudinal magnetic field [10], the present case will also be considered for the sake of simplicity.

The equations for the perturbations are obtained in the usual way by linearizing the general equations of magnetohydrodynamics in the region of the steady state described by Eqs. (1.2). It is convenient to introduce stream functions for the velocity and magnetic field perturbations,

$$u' = \frac{\partial \psi}{\partial y}, \qquad v' = -\frac{\partial \psi}{\partial x},$$

$$B_x' = \frac{\partial \varphi}{\partial y}, \qquad B_y' = -\frac{\partial \varphi}{\partial x}, \tag{2.1}$$

and the quantity $\Theta = T'/T$ in place of the temperature perturbation T'. We thus obtain a system of linearized equations in dimensionless form:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\triangle\psi - U''\frac{\partial \psi}{\partial x} =$$

$$= \frac{1}{R}\left[\eta\triangle\triangle\psi + 2\eta'\triangle\frac{\partial\psi}{\partial y} + \eta''\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right)\right] +$$

$$+ A\left[(\mathbf{B}\nabla)\triangle\varphi - B_x''\frac{\partial\varphi}{\partial x}\right] + \frac{\gamma}{R}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)(\eta U'\Theta), \tag{2.2}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\triangle\varphi - (\mathbf{B}\nabla)\psi =$$

$$= \frac{1}{R_m}\frac{1}{\sigma}\triangle\varphi - \frac{\alpha}{R_m}\frac{B_x'}{\sigma}\Theta, \tag{2.3}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\Theta-(\ln T)'\frac{\partial\psi}{\partial x}=$$
$$=\frac{1}{RP}\left[\varkappa\Delta\Theta+2(1+\beta)\varkappa(\ln T)'\frac{\partial\Theta}{\partial y}\right]+$$
$$+\frac{N}{P}\left[(\gamma-\beta-1)\frac{1}{R}\frac{\eta U'^2}{T}-\right.$$
$$\left.-(\alpha+\beta+1)\frac{A}{R_m}\frac{B_x'^2}{\sigma T}\right]\Theta+$$
$$+2\frac{N}{PR}\frac{\eta U'^2}{T}\left(\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right)+2\frac{NA}{PR_m}\frac{B_x'}{\sigma T}\Delta\varphi$$
$$\left(R=\frac{\rho l v^*}{\eta_0},\ A=\frac{B_0^2}{\mu\rho v^{*2}},\ R_m=\mu\sigma_0 l v^*,\ P=\frac{\eta_0 c_p}{\varkappa_0}\right). \tag{2.4}$$

Here R is the Reynolds number, A is the Alfven number, R_m is the magnetic Reynolds number, P is the Prandtl number, U is the velocity of the unperturbed flow, B is the unperturbed magnetic field, T is the unperturbed temperature, while σ, $\varkappa$, η are the electrical and thermal conductivity and viscosity in the unperturbed flow. The primes denote differentiation with respect to y.

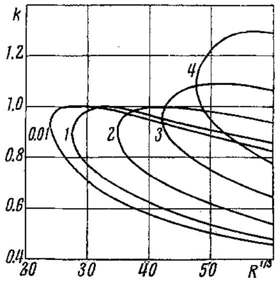

Fig. 1

As usual the solution of the system is written in the form

$$\psi=\psi(y)\exp ik(x-ct), \tag{2.5}$$

where k is the dimensionless wave number and kc is the dimensionless frequency of the oscillations. Equations (2.2)–(2.4) must be solved for the following obvious conditions:

$$\psi(\pm 1)=\psi'(\pm 1)=0, \qquad \Theta(\pm 1)=0. \tag{2.6}$$

The boundary conditions for the magnetic field in the case of nonconducting walls have the form

$$(\varphi'/\varphi)_{\pm 1}=\mp k. \tag{2.7}$$

If system (2.2)–(2.4) is not separable, then hydrodynamic, electrodynamic, and thermal effects exert a simultaneous influence on the stability.

3. The Overheat Instability. We shall first of all consider the case $S \ll R_m$, where $S = M^2/R$ is the hydromagnetic interaction parameter. Clearly in this case field perturbations caused by the motion of the medium may predominate over velocity perturbations caused by the field. In the limit for $A \to 0$ for $\gamma = 0$ we may imagine a situation when the velocity perturbations also tend to zero, and the terms containing ψ in Eqs. (2.3), (2.4) may be neglected. If we make the further assumption that $R_m \ll 1$, then we have from (2.3)

$$\Delta\varphi=\alpha B_x'\Theta. \tag{3.1}$$

Using (2.4), (2.5), and (3.1) and neglecting for simplicity the contribution of viscous dissipation and the fact that $\varkappa$ is not constant, we obtain, after making formal transformations,

$$\Theta''+(E-V)\Theta=0,$$
$$E=-k^2+ikcRP,$$
$$V=-\alpha\Pi\frac{j^2}{\sigma T}+ikURP \qquad \left(\Pi=\frac{j^{*2}l^2}{\sigma_0\varkappa_0 T_0}\right). \tag{3.2}$$

The problem thus becomes one of finding the eigenvalues of the Schrödinger equation with a complex potential V. If the initial steady state is symmetric with respect to y, then it is not hard to see that ReV is a "potential well," and ImV has the form of a hump. The potential may be expanded in a series to give the Schrödinger equation for a harmonic oscillator in the region of the axis of the channel. Having thus ascertained that finite solutions exist [11], we may employ simple approximate methods in order to investigate (3.2). For example in the quasi-classical approximation we replace d/dy by ik_y and obtain the stability criterion immediately (in dimensional form):

$$\varkappa_0 k^2 > \frac{d\ln\sigma}{d\ln T}\frac{j^2}{\sigma T}. \tag{3.3}$$

Formula (3.3) was obtained previously for the general case in paper [7], but the question of the existence of finite solutions was not considered. The presence of the factor α in inequality (3.3) prompts us to call the instability an overheat instability [5, 7]. For simplicity we shall restrict ourselves to considering the case $S \ll R_m \ll 1$ in the quasi-classical approximation. A similar analysis may be carried out without this last restriction.

4. Hydrodynamic Instability. We shall now consider the other limiting case in which the instability is caused by the purely hydrodynamic mechanism of the untwisting of the velocity gradient vortex. It is well known that the onset of hydrodynamic instability occurs for fairly large Reynolds numbers R. We may therefore neglect the small terms in the right-hand side of (2.2), retaining, however, the old derivative. Further we shall confine ourselves to the case $R_m \ll 1$, where we can neglect terms containing B_x compared with B_0. From Eq. (2.3) we have

$$\varphi''-k^2\varphi=-R_m\sigma\psi'+\alpha B_x'\Theta. \tag{4.1}$$

If the hydromagnetic interaction parameter $S \ll 1$, i.e., the Hartmann number is not very large, then we may eliminate φ from (2.2) using (4.1) and, neglecting small terms, finally arrive at a problem which is one of finding the eigenvalues for an Orr–Sommerfeld type equation

$$(U-c)(\psi''-k^2\psi)-U''\psi=\frac{1}{ikR}\eta\psi^{\mathrm{IV}} \tag{4.2}$$

with boundary conditions (2.6). Thus for $R_m \ll 1$, $S \ll 1$, $\alpha S < 1$ the magnetic field and nonisothermal nature of the flow exert an indirect influence on the stability of the motion, altering the velocity profile and introducing a viscosity profile into Eq. (4.2). In order to solve the problem we use the familiar Heisenberg–Lin method [9]. We shall, as usual, confine ourselves to treating even perturbations over the channel halfwidth $(-1,0)$. Two particular solutions, accurate to terms of the order $\sim(kR)^{-1}$, may be obtained from the inviscid equation by an expansion in powers of k^2:

$$\psi_1 = v\{h_0 + k^2 h_2(y) + k^4 h_4(y) + \cdots\},$$
$$\psi_2 = v\{q_1(y) + k^2 q_3(y) + k^4 q_5(y) + \cdots\}$$
$$(v = U - c),$$
$$h_0 = 1, \quad h_{2n+2} = \int_{-1}^{y} \frac{dy}{v^2} \int_{-1}^{y} h_{2n} v^2 dy \quad (n \geqslant 0),$$
$$q_1 = \int_{-1}^{y} \frac{dy}{v^2}, \quad q_{2n+1} = \int_{-1}^{y} \frac{dy}{v^2} \int_{-1}^{y} q_{2n-1} v^2 dy,$$
$$(n \geqslant 1). \tag{4.3}$$

Two more fundamental solutions are found in converging series from the full equation (4.2) and with an accuracy to terms of the order $\sim (kR)^{-1/3}$, have the form

$$\psi_3 = \int_{\infty}^{\zeta} \int_{\infty}^{\xi} \chi^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\chi)^{3/2}\right] d\chi\, d\xi,$$
$$\psi_4 = \int_{-\infty}^{\zeta} \int_{-\infty}^{\xi} \chi^{1/2} H_{1/3}^{(2)}\left[\frac{2}{3}(i\chi)^{3/2}\right] d\chi\, d\xi,$$
$$\zeta = \left(\frac{U_s' kR}{\eta_s}\right)^{1/3}(y - y_s), \quad U_s' = U'(y_s),$$
$$\eta_s = \eta(y_s), \quad U_1' = U'(-1). \tag{4.4}$$

Here $H_{1/3}^{(1,2)}$ are Hankel functions, and y_s is determined from the equation $U(y_s) = c$. Knowing the fundamental system of solutions (4.3), (4.4), it is not

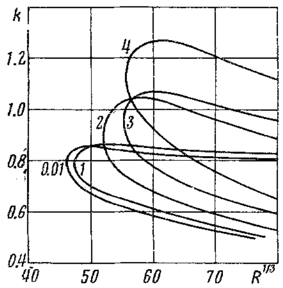

Fig. 2

difficult to obtain the characteristic equation for determining the curves of neutral oscillations

$$F(z) = \frac{(1+\lambda)g}{1+\lambda g} \qquad (g = u^\circ + iv^\circ), \tag{4.5}$$

where $F(z)$ is a tabulated function [9],

$$z = \left(\frac{U_s' kR}{\eta_s}\right)^{1/3}(1 + y_s), \qquad \lambda = (1 + y_s)\frac{U_1'}{c} - 1,$$
$$u^\circ = 1 + U_1' c K_1 + \frac{U_1' c}{k^2} \frac{1 - k^2 N_2 - k^4 N_4 - \cdots}{H_1 + k^2 H_3 + k^4 H_5 + \cdots},$$
$$v^\circ = -\pi c U_1' \frac{U_s''}{U_s'^3}, \quad K_1 = \int_{-1}^{0} \frac{dy}{v^2}, \quad H_1 = \int_{-1}^{0} v^2 dy,$$
$$N_2 = \int_{-1}^{0} v^2 dy \int_{y}^{0} \frac{dy}{v^2}, \quad H_3 = \int_{-1}^{0} v^2 dy \int_{-1}^{y} v^{-2} dy \int_{-1}^{y} v^2 dy,$$

etc. In the calculations, terms of the order $\sim k^2$ were retained, and taking N_4 and H_5 into account improves the accuracy by a few percent.

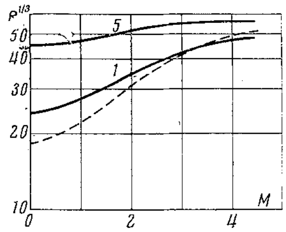

Fig. 3

Equation (4.5) is convenient for numerical calculations. However, even before calculations are performed some qualitative conclusions may be drawn from an analysis of the velocity profile. It was shown by Lock [1] that as the Hartmann number M increases, so also does the critical value of the Reynolds number R_*, i.e., the flow is stabilized. In other words, monotonic, fuller velocity profiles correspond to higher values of R_*. Since the conductivity increases with the temperature in a plasma, the ponderomotive force increases at the center of the stream where the temperature is higher, which leads to an additional flattening of the velocity profile in the case of nonisothermal flow [4]. Thus an increase in R_* may be expected, and the larger the index α and the larger the current flowing in the plasma, the stronger this effect will be.

Figures 1 and 2 present curves of the neutral oscillations for values of $N = 1, 5$, respectively, in the case of a fully ionized plasma ($\alpha = 3/2$, $\beta = \gamma = 5/2$. The numbers on the curves indicate the value of M. It should be noted that for $M > 5$ it is no longer convenient to calculate the curves by the method used here as a result of the worsening of convergence of the expansion in k^2. For the sake of comparison with the results of paper [1], the calculations were carried out for a flow configuration with zero total current. Figure 3 shows $R^{1/3}$ as a function of M for the flow configurations indicated (the dashed line corresponds to the results of [1]). Comparison of the curves leads to the conclusion that additional flow stability results from nonisothermal effects for $M < 5$, while the difference in the values of R decreases as M increases. This is explained by the fact that for a given N the difference in the velocity profiles for isothermal and nonisothermal flow decreases as M increases [4]. It is also worth noting that R as a function of M increases less rapidly the greater N is, so that it is even possible for the curve to intersect that corresponding to the isothermal case. However, special care is required to determine the stability criterion for $M > 5$.

In conclusion we note that in the case of nonisothermal flow of a liquid metal the neutral curves may behave in a radically different way, since the conductivity of the metal decreases as the temperature increases and nonisothermal effects lead to a more extended velocity profile.

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REFERENCES

1. R. C. Lock, The Stability of the Flow of an Electrically Conducting Fluid between Parallel Planes under a Transverse Magnetic Field, Proc. Roy. Soc. A, vol. 233, no. 1192, 1955.

2. B. A. Tverskoi, "The stability of flow of a highly conducting fluid transverse to a magnetic field, Dokl. AN SSSR, vol. 143, no. 2, 1962.

3. J. G. Heywood, An MHD Channel Flow with Temperature-Dependent Electrical Conductivity, AAIA Journal vol. 3, no. 9, 1965.

4. Yu. V. Sanochkin and S. S. Fillipov, "Nonisothermal plasma flow in a flat MHD channel," PMTF

[Journal of applied Mechanics and Techanical Physics], no. 6, 1966.

5. B. B. Kadomtsev, "Hydromagnetic plasma stability," collection: Questions in Plasma Theory, No. 2 [in Russian], Gosatomizdat, 1963.

6. J. K. Wright, A Temperature Instability in Magnetohydrodynamic Flow, Proc. Phys. Soc, vol. 81, pt. 3, no. 521, 1963.

7. Yu. V. Sanochkin, "The dissipative instability in magnetohydrodynamics," Magnitnaya gidrodinamika [Magnetohydrodynamics], no. 3, 1965.

8. Yu. V. Sanochkin, "The overheat plasma instability," Zh. tekh. fiz., vol. 35, no. 6, 1965.

9. Lin Chia-chiao, The Theory of Hydrodynamic Stability [Russian translation], Izd. inostr. lit., 1958.

10. D. H. Michael, The Stability of the Plane-parallel Flow of an Electrically Conductive Fluid, Proc. Cambridge Philos. Soc, vol. 49, p. 1, 1953.

11. A. A. Galeev, "Stability theory of a nonuniform rarified plasma in a strong magnetic field," Zh. Experim. i teor fiz., vol. 44, no. 6, 1963.

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